Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 15, 16

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Textbook Sections: 11.1

Today's Topic: Sequences

Def: A **sequence** is an ordered list of numbers (or, equivalently, a function from the positive integers $\mathbb{N}=\{1,2,3,\dots\} \to \mathbb{R}$). Traditionally, we denote a sequence by writing

$$\{a_n\}_{n=1}^{\infty} \quad \text{or} \quad a_1, a_2, a_3, \dots$$

Here $a_n \in \mathbb{R}$ is the numbers assigned to the positive integer $n \in \mathbb{N}$ (we call a_n the n^{th} term of the sequence).

Often times: We can write down a **formula** for a_n in terms of $n \in \mathbb{N}$:

Examples: $\{a_n\}_{n=1}^{\infty}$ where:

(i) $a_n = n \rightsquigarrow \{1, 2, 3, 4, \dots\}$

(ii) $a_n = n - n^2 \rightsquigarrow \{0, -2, -6, -12, \dots\}$

(iii) $a_n = \sin(n^2) \rightsquigarrow \{\sin(1), \sin(4), \sin(9), \dots\}$

(iv) $a_n = (-1)^n \frac{n+1}{e^n} \rightsquigarrow \{-\frac{2}{e}, \frac{3}{e^2}, -\frac{4}{e^3}, \dots\}$

Many times: We can write down a **recursive formula** for a_n in terms of **prior terms** a_k for $k \in \{1, 2, 3, \dots, n-1\}$:

Example: $\{a_n\}_{n=1}^{\infty}$ where:

(i) $a_1 = 1$ and for $n \geq 2$ $a_n = \frac{a_{n-1}}{2}$:

$$\left\{1, \frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}, \ldots\right\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$$

(ii) $a_1 = 1$, $a_2 = 1$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$:

(Fibonacci) $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

Interesting questions we can ask about sequences:

(i) Does $\{a_n\}_{n=1}^{\infty}$ encode interesting info?

eg) The Fibonacci sequence appears frequently in nature — as but one example, as $n \to \infty$, $\frac{a_{n+1}}{a_n}$ converges to the "golden ratio," and this same ratio appears similarly in the arrangement of seeds in sunflowers, pinecones, the genetic fingerprint of honey bees, etc...

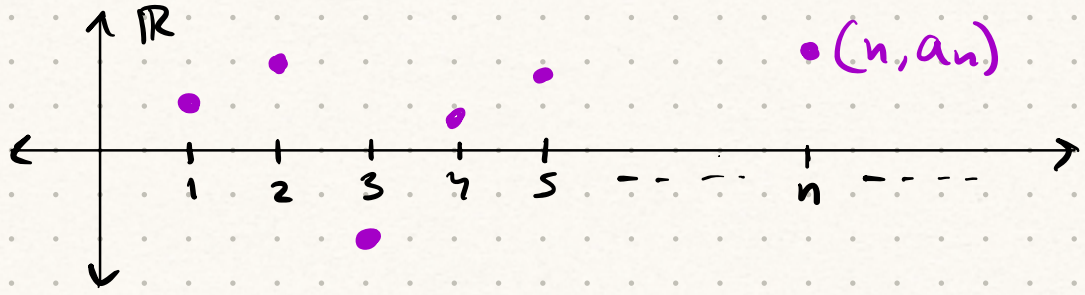
(ii) What is the mathematical behavior of $\{a_n\}_{n=1}^{\infty}$?

eg)
- Do the terms a_n tend to some definite value?
- Do they oscillate?
- Do they wander off to ∞?
- How fast do these occur?

A good way to start thinking about all of this:

Graph the sequence (here it really is helpful to think of $\{a_n\}_{n=1}^{\infty}$ as a function from $\mathbb{N} \to \mathbb{R}$)

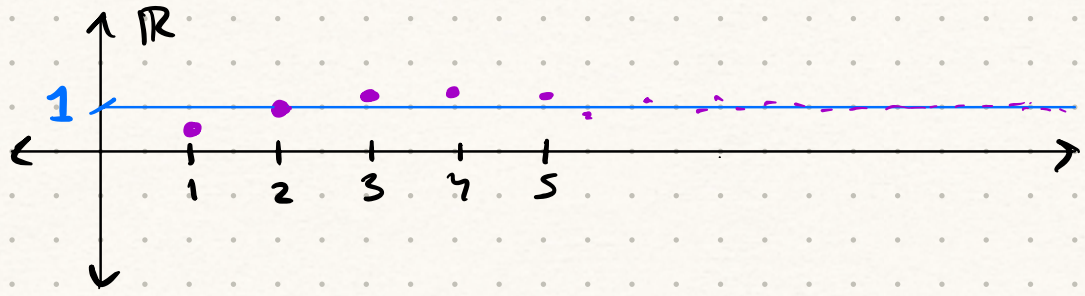
eg:



Here are some example behaviours:

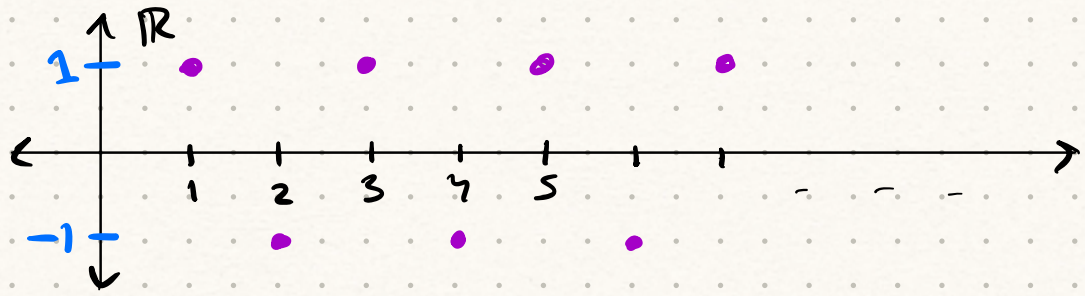
Convergence

$a_n = 1 + \frac{(-1)^n}{n}$



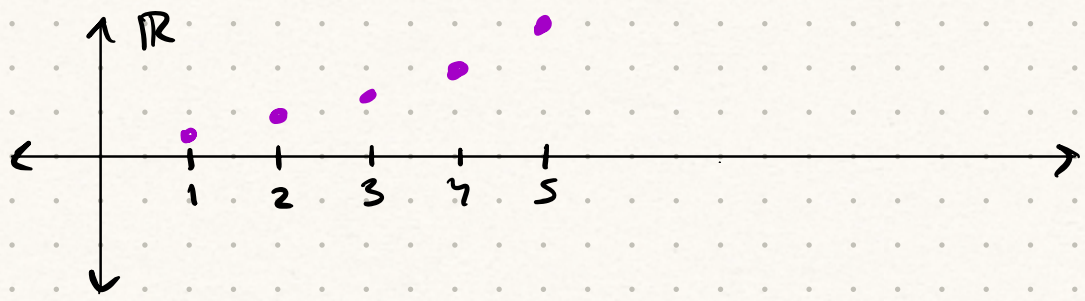
Oscillation:

$a_n = (-1)^n$



Diverge to ∞

$a_n = n$



Let's make this mathematically precise:

Main Def: A sequence $\{a_n\}_1^\infty$ has **limit** $L \in \mathbb{R}$ provided:

For every choice of small number $\varepsilon > 0$ (eg $\varepsilon = \frac{1}{10^{10}}, \frac{1}{10^{100}}, \ldots$) there is a big enough "time" $N \in \mathbb{N}$ so that for every $n \geq N$, (eg $n = 10^{10}, 10^{100}, \ldots$)

$$L - \varepsilon \leq a_n \leq L + \varepsilon.$$

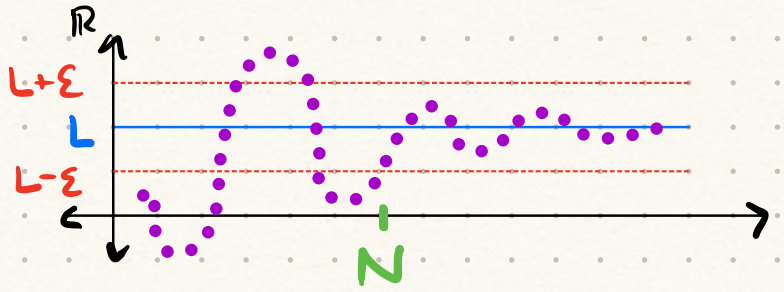
In this case, we write $\lim_{n \to \infty} a_n = L$.

What's this mean?

In words:

a_n is as close to L as we want, as long as we wait long enough (i.e. let n be big enough).

In Pictures



Divergence to ∞: A sequence $\{a_n\}_1^\infty$ **diverges to ∞** provided:

For every choice of huge number $M > 0$ (eg $M = 10^{10}, 10^{100}, \dots$) there is a big enough "time" $N \in \mathbb{N}$ so that for every $n \geq N$ ($n = 10^{10}, 10^{100}, \dots$),

$$a_n \geq M$$

In this case, we write $\lim_{n\to\infty} a_n = \infty$.

Examples:

(1) $\{a_n\}_{n=1}^\infty$, $a_n = 1 + \frac{(-1)^n}{n}$

Claim: $\lim_{n\to\infty} a_n = 1$.

Proof: Fix any tiny number $\varepsilon > 0$. If n is bigger than $\frac{1}{\varepsilon}$, then we have

$$|a_n - 1| = \left|\left(1 + \tfrac{(-1)^n}{n}\right) - 1\right| = \left|\tfrac{(-1)^n}{n}\right| = \tfrac{1}{n} < \varepsilon$$

so a_n is always trapped ε close to 1 if $N > \frac{1}{\varepsilon}$.

(2) $\{b_n\}_{n=1}^\infty$, $b_n = \frac{n+1}{\sqrt{n}}$

Claim: $\lim_{n\to\infty} a_n = \infty$

Proof: Fix any huge number $M > 0$. If n is bigger than M^2, then.

$$a_n = \frac{n+1}{\sqrt{n}} \geq \frac{n}{\sqrt{n}} = \sqrt{n} > M.$$

(3) $\{c_n\}_{n=1}^\infty$ $\quad c_n = (-1)^n$

Claim: $\{c_n\}$ has no limit.

Proof: If it did, say $\lim_{n\to\infty} c_n = C$, then for any $\varepsilon > 0$ (say $\varepsilon = \frac{1}{1000}$) we would have $C - \varepsilon \leq c_n \leq C + \varepsilon$ for all n large. But $|c_n - c_{n+1}| = 2 > \frac{2}{1000}$. . . .

Part II: Lecture 16

Today's Goal: Define and explore limits of sequences

⚠ First, go back to the **Main Defn** of **Lecture 15** and start there.

Limit Laws: What follows is a collection of rules for doing computations with limits of sequences:

Setup: Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit A

Let $\{b_n\}_{n=1}^{\infty}$ be " " B

Let $c \in \mathbb{R}$ be a fixed number

1) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

2) $\lim_{n\to\infty} (c \cdot a_n) = c \cdot A$

3) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

4) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ if $B \neq 0$.

5) $\lim_{n\to\infty} f(a_n) = f(A)$ if $f(x)$ is any function continuous at A.

6) If $\lim_{x\to\infty} f(x) = L$, and we set $a_n = f(n)$ for $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} a_n = L$.

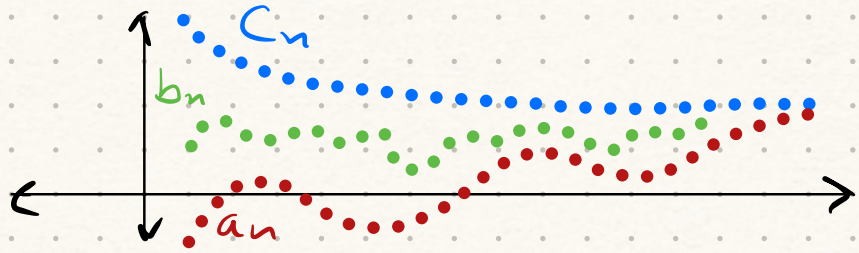
7) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

We'll get practice with a few more examples in just a moment, but before we do, here's an extremely useful theorem that is very helpful in proving the existence of certain limits:

Squeeze Theorem Suppose $a_n \le b_n \le c_n$ for all $n \ge 1$.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ too.

In pictures:



Examples:

(i) $a_n = \frac{n}{n+1}$ $\quad a_n = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$. $1+\frac{1}{n} \to 1$. $a_n \to \frac{1}{1} = 1$.

(ii) $b_n = \sin\left(\frac{\pi}{n}\right)$ $\quad \frac{\pi}{n} \to 0$, $\sin\left(\frac{\pi}{n}\right) \to \sin(0) = 0$.

(iii) $c_n = \frac{2^n}{n!}$ $\quad 0 < c_n = \frac{2\cdot 2 \cdots 2\cdot 2}{n\cdot(n-1)\cdots 2\cdot 1} \le \frac{2}{n}\cdot\left(\frac{2\cdots 2\cdot 2}{(n-1)\cdots 2\cdot 1}\right) \le \frac{2}{n}\cdot 2$

(iv) **Challenge Problem**: For exactly which values of $r \in \mathbb{R}$ is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Lastly for today: There are certain types of sequences (which appear quite a bit in practice) which are guaranteed to converge

Def: i) If $\{a_n\}_{n=1}^{\infty}$ is such that $a_{n+1} \overset{(\le)}{\ge} a_n$ for all n, we say $\{a_n\}_{n=1}^{\infty}$ is monotonically increasing (decreasing) (or in general, monotone)

ii) If $a_n \overset{(\ge)}{\le} M$ for all $n \ge 1$, we say that $\{a_n\}_{n=1}^{\infty}$ is bounded from above (from below) (if both, we say $\{a_n\}$ is bounded).

Theorem (Monotone Convergence)

Any sequence which is both monotone and bounded is convergent, and has a finite limit.

Example: $\{a_n\}_{n=1}^{\infty}$, $a_1 = 2$ $\quad a_{n+1} = \frac{1}{2}(a_n + 6)$.

First, we can show $\{a_n\}$ is monotone increasing:

For $n=1$: $a_2 = \frac{1}{2}(a_1 + 6) = \frac{1}{2}(2+6) = 4 \geq a_1 = 2$.

Now suppose $a_{k+1} \geq a_k$ for $k \geq 1$. Let's show $a_{k+2} \geq a_{k+1}$ as well:

$$a_{k+2} = \frac{1}{2}(a_{k+1} + 6) \geq \frac{1}{2}(a_k + 6) = a_{k+1}$$

so, since the bound is true for $k=1$ $(a_2 \geq a_1)$ and we just showed that $a_{k+1} \geq a_k$ implies $a_{k+2} \geq a_{k+1}$, we see that $a_3 \geq a_2$, $a_4 \geq a_3$, ..., and so on. In particular, $a_{n+1} \geq a_n$ for all n, so $\{a_n\}$ is monotone increasing.

Next, we can show $\{a_n\}$ is bounded above:

Of course, $a_1 = 2 \leq 6$.

Suppose that for some $k \geq 1$ $\quad a_k \leq 6$. We show that then $a_{k+1} \leq 6$ too. Indeed,

$$a_{k+1} = \frac{1}{2}(a_k + 6) \leq \frac{1}{2}(6+6) = 6.$$

So, by "induction" as before, $a_n \leq 6$ for all n.

Thus, $\{a_n\}_{n=1}^{\infty}$ must have a limit by the Theorem.

Lastly, we can compute the limit: Let $L = \lim_{n\to\infty} a_n$.

Then

$$L = \lim_{n\to\infty} a_{n+1} = \frac{1}{2}\lim_{n\to\infty}(a_n + 6) = \frac{1}{2}(L+6)$$

$$\Rightarrow L = 6$$